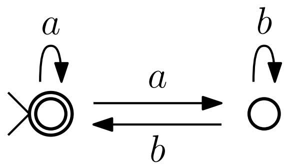
COSC 341 – Tutorial 7 (Solution)

1. Are the following languages automatic languages? If so, construct an NFA for that language. If not, prove that the language is not automatic.

   (a) $L = \{w |$ in w every a is followed by a $b\}$
   L is automatic since we can define an NFA that accepts L:

   (b) $L = \{w |$ for every a in w there is a distinct b following $a\}$
   We will prove that L is not automatic.
   For contradiction we assume that L is automatic.
   Let us consider $z = a^k b^k \in L, |z| \geq k$. Because of the Pumping Lemma there are u, v and w such that $z = uvw$ with $|u| + |v| \leq k,\ |v| > 0$, and $uv^i w \in L$ for all $i \geq 0$. From $|u| + |v| \leq k,\ |v| > 0$ we can follow that $u = a^{|u|}, v = a^{|v|}$, and $w = a^{k-(|u|+|v|)} b^k$.
   By the Pumping Lemma, uv^2w is element of L as well. It is

   $$uv^2w = a^{|u|} a^{2|v|} a^{k-(|u|+|v|)} b^k = a^{k+|v|} b^k$$

   with $|v| > 0$. Therefore, uv^2w being an element of L is a contradiction to the definition of L.
   We conclude that L is not an automatic language.

   (c) $L = \{a^i | i \text{ is prime}\}$
   We will prove that L is not automatic.
   For contradiction we assume that L is automatic.
   Because of the Pumping Lemma we know that there is a $k > 0$ such that if $z \in L, |z| \geq k$, there are u, v and w such that $z = uvw$ with $|u| + |v| \leq k,\ |v| > 0$, and $uv^i w \in L$ for all $i \geq 0$.
   Let n be a prime number greater than k. We can apply the Pumping Lemma on $z = a^n$ and know that there are u, v, w with the properties stated above.
   In particular, the Pumping Lemma implies that $uv^{n+1}w \in L$. But it is

   $$\begin{aligned} length(uv^{n+1}w) &= length(uvv^n w) \\ &= length(uvw) + length(v^n) \\ &= n + length(v) \cdot n \\ &= n(1 + length(v)) \end{aligned}$$

   This implies that $uv^{n+1}w$ is not prime which is a contradiction to the definition of L.
   We conclude that L is not an automatic language.

   (d) $L = \{a^n b^{n+1} \mid n \geq 0\}$
   We will prove that L is not automatic.
   For contradiction we assume that L is automatic.
   Let us consider $z = a^k b^{k+1} \in L, |z| \geq k$. Because of the Pumping Lemma there are u, v and w such that $z = uvw$ with $|u| + |v| \leq k,\ |v| > 0$, and $uv^i w \in L$ for all $i \geq 0$. From $|u| + |v| \leq k,\ |v| > 0$ we can follow that $u = a^{|u|}, v = a^{|v|}$, and $w = a^{k-(|u|+|v|)} b^{k+1}$.
   By the Pumping Lemma, uv^2w is element of L as well. It is

   $$uv^2w = a^{|u|} a^{2|v|} a^{k-(|u|+|v|)} b^{k+1} = a^{k+|v|} b^{k+1}$$

   with $|v| > 0$. Therefore, uv^2w being an element of L is a contradiction to the definition of L.
   We conclude that L is not an automatic language.

Homework

1. Are the following languages automatic languages? If so, construct an NFA for that language. If not, prove that the language is not automatic.

   (a) $L = \{a^i | i = n^2, n \in \mathbb{N}\}$
   We will prove that L is not automatic.
   For contradiction we assume that L is automatic.
   Because of the Pumping Lemma we know that there is a $k > 0$ such that if $z \in L, |z| \geq k$, there are u, v and w such that $z = uvw$ with $|u| + |v| \leq k,\ |v| > 0$, and $uv^iw \in L$ for all $i \geq 0$.
   We can apply the Pumping Lemma on $z = a^{k^2}$ and know that there are u, v, w with the properties stated above.
   In particular, the Pumping Lemma implies that $uv^2w \in L$. But it is

$$\begin{aligned} length(uv^2w) &= length(uvw) + length(v) \\ &= k^2 + length(v) \\ &\leq k^2 + k \\ &< k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

   Hence, it is $k^2 < length(uv^2w) < (k+1)^2$. So we can not find an $n \in \mathbb{N}$ such that $length(uv^2w) = a^{n^2}$, which is a contradiction to $uv^2w \in L$.
   We conclude that L is not an automatic language.

   (b) $L = \{w | w \in \{a, b\}^*,$ the total number of a's and b's in w is divisible by $3\}$
   L is automatic since we can define an NFA that accepts L: